BMO1

Reformulating the Equation

BMO1 takes the following form. Start with the pair of points $(1, 2)$, and iterate them under the following map.

$$(x_{n+1}, y_{n+1}) = \begin{cases} (x_n - y_n, 4y_n + 2) \text{ if } x_n > y_n \\ (2x_n + 1, y_n - x_n) \text{ if } y_n > x_n \end{cases}$$

The machine halts if and only if x_n ever equals y_n for some n.

We can rewrite homogenously:

$$(x_{n+1}, y_{n+1}, z_{n+1}) = \begin{cases} (x_n - y_n, 4y_n + 2z_n, z_n) \text{ if } x_n > y_n \\ (2x_n + z_n, y_n - x_n, z_n) \text{ if } y_n > x_n \end{cases}$$

Then, we can divide out by a common factor in such a way that $x_n + y_n + z_n = 1$.

Our new starting point is:

$$(x_0, y_0, z_0) = (1/4, 1/2, 1/4)$$

$$(x_{n+1}, y_{n+1}, z_{n+1}) = \begin{cases} (\frac{x_n - y_n}{1+2y_n+2z_n}, \frac{4y_n+2z_n}{1+2y_n+2z_n}, \frac{z_n}{1+2y_n+2z_n}) \text{ if } x_n > y_n \\ (\frac{2x_n+z_n}{1+z_n}, \frac{y_n-x_n}{1+z_n}, \frac{z_n}{1+z_n}) \text{ if } y_n > x_n \end{cases}$$

(We have used the fact that $x_n + y_n + z_n = 1$ to simplify the denominators in the above expression.)

Finally, set $z_n = 1 - x_n - y_n$, to return to a 2 variable equation:

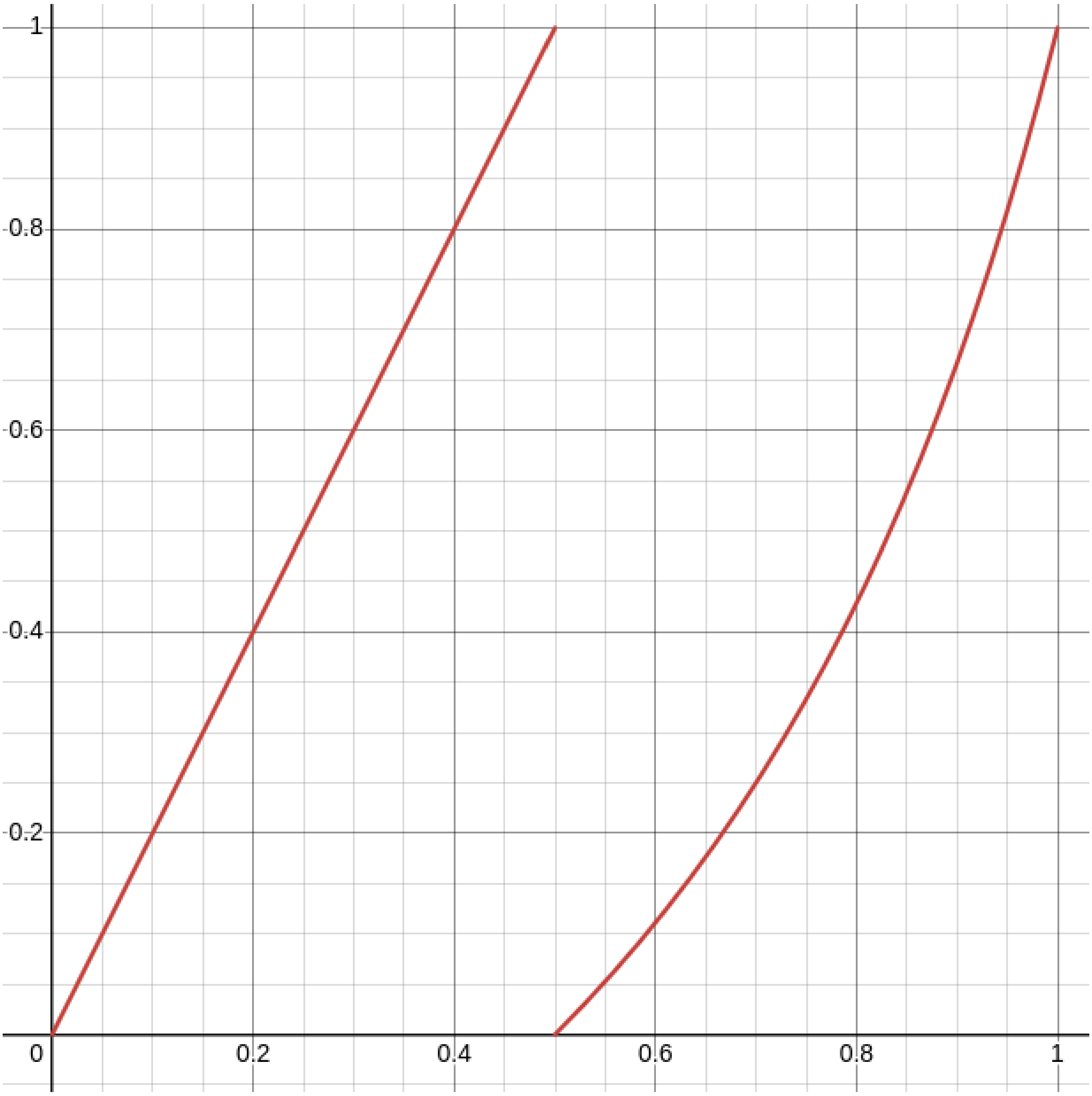
$$(x_{n+1}, y_{n+1}) = \begin{cases} (\frac{x_n - y_n}{-2x_n+3}, \frac{-2x_n+2y_n+2}{-2x_n+3}) \text{ if } x_n > y_n \\ (\frac{x_n-y_n+1}{-x_n-y_n+2}, \frac{y_n-x_n}{-x_n-y_n+2}) \text{ if } y_n > x_n \end{cases}$$

1D Limit

Since $z_n = 1 - x_n - y_n$ is always decreasing in size (and it's limit as n tends to infinity must be 0), we will analyse the limiting behaviour when $z_n = 0$, or $y_n = 1 - x_n$. This limit is a 1-dimensional.

$$x_{n+1} = \begin{cases} \frac{2x_n-1}{-2x_n+3} \text{ if } x_n > \frac{1}{2} \\ 2x_n \text{ if } x_n < \frac{1}{2} \end{cases}$$

A map of $f(x_n)$:



If the right hand side were simply $2x_n - 1$, then the points that reached $\frac{1}{2}$ would simply be the dyadic rationals in the range $(0, 1)$, but it is unfortunately more complicated.

Density of solutions

One hope for proving BMO1 might be the following: Try and prove that the points that reach $\frac{1}{2}$ in our $z = 0$ limit are not dense, but that there are regions of open sets which have no converging points in them. Call the union of open sets which have no solutions in them U. Then, prove that our starting case, in the limit of $z \to 0$, will approach a region which is in U.

Unfortunately, this is impossible. The solutions are dense. To prove this, we will use a proof by contradiction:
Suppose there is an open interval (a_0, b_0), with size $L_0 = b_0 - a_0$, with no points that will eventually reach $\frac{1}{2}$ under iteration on f. First, note that the derivative of both functions in f are always greater than 1 over their respective ranges, so therefore, applying the function on the range (a_0, b_0) to get a new interval over the range (a_1, b_1) (the only way the interval could be split is if it contained the point $\frac{1}{2}$, which breaks our assumption), the new length L_1 will be

greater than the old by a factor of $\alpha > 1$ (or, more strongly, $L_{n+2} \geq 2L_n$). Repeatedly applying this result, L_n must eventually become larger than $\frac{1}{2}$, implying it covers the point $\frac{1}{2}$. This is a contradiction from our starting assumption, so every interval (a, b) must have a point that reaches $\frac{1}{2}$, and so the terminating points in f must be dense over $(0, 1)$.

Fast algorithm for proving a point halts or loops

Given a point $x \in (0, 1)$, we want to know if x ever reaches $\frac{1}{2}$. It is easy to see that for any irrational number, it will never reach $\frac{1}{2}$, since it will remain irrational under each iteration of our map, and so never reach a rational value. So we will restrict our attention to the case of $x_n = \frac{u_n}{v_n}$. The iteration function for fractions looks like

$$(u_{n+1}, v_{n+1}) = \begin{cases} (2u_n - v_n, -2u_n + 3v_n) \text{ if } 2u_n > v_n \\ (2u_n, v_n) \text{ if } 2u_n < v_n \end{cases}$$

Note that in practice, you would likely simplify after each step so that $\gcd(u_n, v_n) = 1$.
Lemma: If v_n (in simplified form) is odd, then v_{n+1} (in simplified form) is also odd.

- If the first branch is applied, v_{n+1} is odd, because it's an even term $-2u_n$, plus an odd term ($3v_n$, two odd terms multiplied together). Adding them together produces an odd term.
- In the second case, v_{n+1} is odd, because it is equal to v_n which is odd.
- This will remain true even if a common factor is factored out from u_n and v_n, since a factor of an odd term is never even.
  Also note that the termination condition occurs when $\frac{u_n}{v_n} = \frac{1}{2}$, or $2u_n = v_n$, which requires v_n to be even. As a result, if v_n is ever odd, we know it will never halt.

This allows us to find a write the following program to prove a rational point in the range $(0, 1)$ halts or loops:

```
def BMO1_1D_halts(frac:Fraction):
    assert(frac>0 and frac<1)
    while True:
        if frac==0.5:
            return True
        if frac.denominator%2==1:
            #denominator odd, so denominator will always be odd
            #so never =1/2, so loops
            return False
        frac=BMO1_1D_step(frac)
```

Strictly speaking, I have not ruled out the possibility of a cycle, which would cause the above algorithm to loop, but this empirically does not happen for any rational point with denominator < 5000, so I conjecture that there are no such cycling points with even denominator (note this conjecture is almost certainly not needed to prove or disprove BMO1). I also have not ruled out the possibility that distinct terms with even denominators could not keep appearing forever, but probabilistic arguments demonstrate that this is highly unlikely.

Returning to the full 2D case

We'd like to take what we've learned about the 1D limit case, and apply it to the 2D case. First, consider the following lemma.
Lemma: For any given starting point (x, y), there is a point α in our 1D case that has the same trajectory.

By 'same trajectory', we mean that if our 2D point (x, y) takes the first branch in our iterated function, then α will take the first branch in the 1D iterated function. Similarly, if (x, y) takes the second branch, so will α. Finally, if (x, y) halts, so will α. This is true not just for a single iterations, but for all iterations until either both programs halt, or to infinite time.

The proof is simply that every possible trajectory occurs for some $x \in [0, 1]$, which follows from the fact that both branches map from either $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$ to $(0, 1)$, so any trajectory of length $n - 1$ can be extended to a trajectory of length n, by prepending a starting value that maps onto the starting value of the length $n - 1$ trajectory, from the first or second maps respectively. Proceeding by induction, we can find a trajectory for all finite lengths n.

So, if we want to prove $(\frac{1}{4}, \frac{1}{2})$, our starting point, loops forever, we need to find the equivalent value α, and show either that:

- α is irrational, or
- α is rational, but is a looping point according to the algorithm described above.

To start with, let's look for a rational point that matches the trajectory of $(\frac{1}{4}, \frac{1}{2})$ for all time. An efficient algorithm that will find such a rational point in finite time if it exists as follows:

- Set an initial lower bound of 0
- Set an initial upper bound of 1
- Set our test fraction to be the mediant of your upper and lower bounds (essentially, the simplest fraction between the lower and upper bounds, found by taking $mediant = \frac{lower\_numerator + upper\_numerator}{lower\_denominator + upper\_denominator}$), and test the trajectory of the mediant against that our our starting point $(\frac{1}{4}, \frac{1}{2})$.
  - If the mediant takes the second path, while our starting point takes the first path, then the mediant is too big, so set it to be our new upper bound
  - If the mediant takes the first path, while our starting point takes the second path, then the mediant is too small, so set it to be our new lower bound.
- Recalculate the mediant using our new lower and upper bounds, and run the comparison algorithm again. Continue until your denominator is above a cutoff, or you find a rational point which matches the value.

In the case of the starting point $(\frac{1}{4}, \frac{1}{2})$, we did not find a rational function which matched the path for all time. The value it converged to is approximately 0.3621250257705563..., and if it has any rational approximation, it's denominator is very large ($> 10^{50}$). As such, I conjecture that this value is irrational. Proving this result would be enough to prove BMO1 loops. On the other hand, if it were rational and an explicit form were given, it could be passed to the function given above, which would be enough to demonstrate whether the point halts or loops.