Macro analysis of a BB(6) cryptid

Introduction

We study a 6-state Turing machine candidate ("cryptid") for the Busy Beaver problem. After a standard acceleration step, the machine's behaviour reduces to a macro dynamical system on three kinds of states: $P(a)$, $Q(a,b)$, and Halt. The machine halts when (and only when) the orbit reaches $Q(0,b)$ for some b. This can happen in two independent ways, depending on which Q-entry value the orbit produces:

- **Direct halt.** Q-entry values in $\{2^{i+1}-2 : i \geq 1\} = \{2,6,14,30,\ldots\}$ give rise to halving chains $Q(2^{i+1}-2,\_) \to \cdots \to Q(2,\_) \to Q(0,\_) = \mathsf{Halt}$ that reach $Q(0,\_)$ without passing through $Q(1,\_)$.
- **Via $Q(1,\_)$.** Q-entry values in $\{3\cdot 2^k-2 : k \geq 1\} = \{4,10,22,46,\ldots\}$ give rise to halving chains that pass through $Q(1,\_)$, triggering rules Q1/QP1 and re-entering the P-phase with new values.

The present note focuses on the second question: the reachability of $Q(1,\_)$ from the initial state $P(2)$. We develop the formal theory (verified in Lean 4 with Mathlib), giving closed forms for the P-phase map, characterising the Q-phase accumulator, and proving that $Q(1,\_)$-reachability is equivalent to a Collatz-like Diophantine condition on the orbit.

1 Macro transition rules

The macro map $\mathsf{next}\colon \mathsf{State} \to \mathsf{State}$ acts by:

$$\begin{array}{lll}
\textbf{PE:} & P(2a) & \longmapsto P(3a+4), \\
\textbf{PQ:} & P(2a{+}1) & \longmapsto Q(a{+}2,\ 1), \\[4pt]
\textbf{H:} & Q(0,b) & \longmapsto \mathsf{Halt}, \\
\textbf{Q1:} & Q(1,2b) & \longmapsto Q(b{+}2,\ 1), \\
\textbf{QP1:} & Q(1,2b{+}1) & \longmapsto P(3b{+}8), \\
\textbf{Q2:} & Q(2a{+}2,b) & \longmapsto Q(a,\ b{+}2a{+}5), \\
\textbf{QP2:} & Q(2a{+}3,b) & \longmapsto P(b{+}5a{+}6).
\end{array}$$

The naming convention: rules Q1, QP1 are the two cases with first argument 1; rules Q2, QP2 are the halving/exit rules with first argument ≥ 2. The orbit starts at $P(2)$.

For a state s, we write $\mathsf{next}^n(s)$ for the n-fold iterate.

Lemma 1.1 (Right-unfolding)**.** *For every state s and $n \in \mathbb{N}$,*

$$\mathsf{next}^{n+1}(s) = \mathsf{next}\big(\mathsf{next}^n(s)\big).$$

Proof. Induction on n. The base case is immediate; the step follows from the left-recursive definition of iteration: $\mathsf{next}^{n+2}(s) = \mathsf{next}^{n+1}(\mathsf{next}(s)) = \mathsf{next}(\mathsf{next}^{n+1}(s))$ by the inductive hypothesis applied to $\mathsf{next}(s)$. $\square$

2 P-phase recurrence and closed form

When the orbit is in a P-state, rule PE applies repeatedly (each application replaces n by $3n/2+4$, stripping a factor of 2) until the argument becomes odd, at which point rule PQ fires and the orbit enters the Q-phase. The number of PE-steps equals the 2-adic valuation of the argument shifted by 8.

Definition 2.1 (P-phase map)**.** Define $p\colon \mathbb{N}_{\geq 1} \to \mathbb{N}$ by

$$p(2k+1) = k+2, \qquad p(2k) = p(3k+4) \quad (k \geq 1).$$

This recurrence terminates because $v_2(3k+4+8) < v_2(2k+8)$, where $v_2(n)$ =2-adic valuation of n.

Lemma 2.2 (Recurrence cases)**.** *For all $k \geq 1$: $p(2k) = p(3k+4)$. For all $k \geq 0$: $p(2k+1) = k+2$.*

Proof. Both follow by unfolding the definition of p and simplifying the parity check. $\square$

The closed form expresses $p(n)$ in terms of the odd part $\mathrm{ord}_2(n+8) := (n+8)/2^{v_2(n+8)}$ and the 2-adic valuation $v_2(n+8)$.

Theorem 2.3 (Closed form for p)**.** *For $n \geq 1$,*

$$p(n) = \frac{\mathrm{ord}_2(n+8) \cdot 3^{v_2(n+8)} - 5}{2} = \frac{(n+8) \cdot 3^{v_2(n+8)} \,/\, 2^{v_2(n+8)} - 5}{2}.$$

Proof. By strong induction on $v := v_2(n+8)$.

Base ($v = 0$): Then n is odd, say $n = 2k+1$, and $p(2k+1) = k+2$. Since $v_2(2k+9) = 0$, the odd part is $2k+9$ itself, and $(2k+9) \cdot 3^0 = 2k+9$, so the formula gives $(2k+9-5)/2 = k+2$.

Step ($v \to v+1$): Then n is even, $n = 2k$, and $p(2k) = p(3k+4)$ by the recurrence. The key identities are:

$$\begin{aligned} v_2(3k+12) &= v_2(k+4), & \mathrm{ord}_2(3k+12) &= 3 \cdot \mathrm{ord}_2(k+4), \\ v_2(2k+8) &= v_2(k+4)+1, & \mathrm{ord}_2(2k+8) &= \mathrm{ord}_2(k+4). \end{aligned}$$

These follow from the multiplicativity of the 2-adic valuation on $3 \cdot (k+4)$ and $2 \cdot (k+4)$ (with $\gcd(3,2) = 1$). Substituting into the inductive formula for $p(3k+4)$ and simplifying yields the formula for $p(2k)$. $\square$

Corollary 2.4 (Defining equations by 2-adic valuation)**.** *Writing $n+8 = 2^v m$ with m odd, the first four cases are:*

$$\begin{aligned} p(n) &= \frac{n+3}{2}, && v = 0 \ (\textit{odd } n), \\ p(n) &= \frac{3n/2+7}{2}, && v = 1 \ (n{+}8 = 2m,\ m \textit{ odd}), \\ p(n) &= \frac{9n/4+13}{2}, && v = 2 \ (n{+}8 = 4m,\ m \textit{ odd}), \\ p(n) &= \frac{27n/8+22}{2}, && v = 3 \ (n{+}8 = 8m,\ m \textit{ odd}). \end{aligned}$$

The general pattern is $p(n) = (3^v m - 5)/2$.

Proof. Each case follows by applying $p(2k) = p(3k+4)$ exactly v times (stripping all factors of 2 from n) until the argument becomes odd, then applying $p(2k+1) = k+2$. $\square$

Corollary 2.5 (Halting Diophantine equations)**.** *Setting* $p(n) = 2^i - 2$ *(a direct-halting Q-entry value) in each case of Corollary 2.4 yields:*

$$\begin{aligned} v = 0: &\quad n + 7 = 2^{i+1}, && n \in \{1, 9, 25, 57, 121, 249, \ldots\}, && (1)\\ v = 1: &\quad 3n + 22 = 2^{i+2}, && n \in \{14, 78, 334, 1358, 5454, \ldots\}, && (2)\\ v = 2: &\quad 9n + 68 = 2^{i+3}, && n \in \{220, 14\,556, 932\,060, \ldots\}. && (3) \end{aligned}$$

Proof. Clear the denominator and rearrange. For instance, in case $v = 1$: $(3n/2 + 7)/2 = 2^i - 2$ gives $3n/2 + 7 = 2^{i+1} - 4$, hence $3n + 22 = 2^{i+2}$. $\square$

Corollary 2.6 (Image of p)**.** *For every* $n \geq 1$ *there exist* $\ell, k \in \mathbb{N}$ *such that*

$$p(n) = \frac{(2\ell + 1) \cdot 3^k - 5}{2}.$$

Proof. Take $\ell = \lfloor \mathrm{ord}_2(n+8)/2 \rfloor$ and $k = v_2(n+8)$. The odd part is always odd, so it has the form $2\ell + 1$. $\square$

Lemma 2.7 (Connection to macro model)**.** *For every* $n \geq 1$ *there exist* $s \geq 0$ *such that*

$$\mathsf{next}^{s+1}(P(n)) = Q(p(n),\, 1).$$

That is, iterating the macro map from $P(n)$ *through the P-phase terminates at* $Q(p(n), 1)$*.*

Proof. Induction on $v = v_2(n+8)$. If n is odd, a single PQ step gives $Q(\lfloor n/2 \rfloor + 2, 1) = Q(p(n), 1)$. If n is even, one PE step maps $P(n)$ to $P(3n/2 + 4)$, and the 2-adic valuation of the shifted argument decreases by 1, so the inductive hypothesis applies. $\square$

3 Q-phase accumulator

Once the orbit enters the Q-phase at $Q(a, 1)$, rules Q2 and QP2 apply repeatedly (the first argument halves at each Q2 step, accumulating into the second argument) until the first argument becomes odd, at which point QP2 fires and returns to P. The net effect on the second argument is captured by a one-parameter accumulator.

Definition 3.1 (Q-phase accumulator)**.** Define $\mathrm{qacc} \colon \mathbb{N} \to \mathbb{N}$ by

$$\mathrm{qacc}(0) = 0, \quad \mathrm{qacc}(2k) = 2k + 3 + \mathrm{qacc}(k-1), \quad \mathrm{qacc}(2k+1) = 5k + 1.$$

Lemma 3.2 (Recurrence cases for qacc)**.** $\mathrm{qacc}(0) = 0$. *For* $k \geq 1$*:* $\mathrm{qacc}(2k) = 2k + 3 + \mathrm{qacc}(k-1)$. *For* $k \geq 0$*:* $\mathrm{qacc}(2k+1) = 5k + 1$.

Definition 3.3 (Two-argument Q-phase function)**.** $q(a, b) := b + \mathrm{qacc}(a)$.

Remark 3.4. The accumulator qacc follows only the Q2/QP2 chain. It is well-defined for all $a \geq 0$, but at $a = 1$ it gives $\mathrm{qacc}(1) = 1$ whereas the actual Q-phase from $Q(1, b)$ would trigger rule Q1 or QP1 instead of Q2. This discrepancy is irrelevant if $Q(1, \_)$ is never reached—which is the main conjecture.

Closed form for qacc

The recursion $\operatorname{qacc}(2m{+}2) = \operatorname{qacc}(m) + 2m + 5$ (the even case, reindexed via $k = m{+}1$) telescopes along the chain

$$n_0 = n, \quad n_{i+1} = \frac{n_i - 2}{2}, \quad i = 0, 1, \ldots, r{-}1,$$

which terminates when n_r is odd or zero. One checks by induction that $n_i = 2^{r-i}(n_r + 2) - 2$, so $n + 2 = 2^r(n_r + 2)$. Writing $v := v_2(n + 2)$ and $\operatorname{op} := \operatorname{ord}_2(n + 2)$ (the odd part), two cases arise:

- **$\operatorname{op} \geq 3$:** The chain makes $r = v$ steps, landing on odd $n_r = \operatorname{op} - 2 \geq 1$. Summing $\operatorname{qacc}(2m{+}2) = \operatorname{qacc}(m) + 2m + 5$ along the chain and using $\operatorname{qacc}(n_r) = (5n_r - 3)/2$:

$$\begin{aligned}\operatorname{qacc}(n) &= \sum_{i=0}^{v-1}(n_i + 3) + \operatorname{qacc}(n_v)\\ &= \operatorname{op} \cdot 2(2^v - 1) + v + \tfrac{5(\operatorname{op}-2)-3}{2}\\ &= 2n + v + \tfrac{\operatorname{op}-5}{2}.\end{aligned}$$

- $\operatorname{op} = 1$ (i.e., $n{+}2 = 2^v$): The chain makes $r = v{-}1$ steps, landing on $n_r = 0$ with $\operatorname{qacc}(0) = 0$.

$$\operatorname{qacc}(n) = 2 \cdot 2(2^{v-1} - 1) + (v - 1) = 2^{v+1} + v - 5 = 2n + v - 1.$$

These two cases unify into a single formula over $\mathbb{N}$ that avoids negative intermediate values:

Theorem 3.5 (Closed form for qacc)**.** *For $n \geq 1$,*

$$2\operatorname{qacc}(n) + 5 \;=\; 4n + 2\, v_2(n + 2) + \max\bigl(\operatorname{ord}_2(n + 2),\, 3\bigr).$$

Proof. By strong induction on n, following the telescoping argument above. The base case n odd: $\operatorname{qacc}(2k{+}1) = 5k + 1$, and $\operatorname{ord}_2(2k{+}3) = 2k{+}3 \geq 3$, $v_2(2k{+}3) = 0$, so the right side is $8k{+}4 + 0 + (2k{+}3) = 10k{+}7 = 2(5k{+}1) + 5$. For $n = 2m$ even with $m \geq 1$: apply $\operatorname{qacc}(2m) = 2m + 3 + \operatorname{qacc}(m{-}1)$, use $v_2(2m{+}2) = v_2(m{+}1) + 1$ and $\operatorname{ord}_2(2m{+}2) = \operatorname{ord}_2(m{+}1)$, and invoke the inductive hypothesis for $\operatorname{qacc}(m{-}1)$ (noting that $(m{-}1) + 2 = m + 1$). ☐

We call a Q-entry value *forbidden* if the Q2-halving chain starting from it passes through $Q(1, \_)$. Since Q2 halves the first argument at each step, the chain from $Q(a, b)$ reaches $Q(1, \_)$ precisely when a belongs to the backward orbit of 4 under doubling-plus-two, i.e., when $a \in \{3 \cdot 2^k - 2 : k \geq 1\} = \{4, 10, 22, 46, 94, \ldots\}$. Thus a Q-entry $Q(a, b)$ is forbidden if and only if $a = 3 \cdot 2^k - 2$ for some $k \geq 1$.

4 Backward reachability of $Q(1, \_)$

The central question is whether the orbit from $P(2)$ ever reaches a state $Q(1, b)$. If it does not, then rules Q1 and QP1 are dead code, the Q-phase is always governed by Q2/QP2, and the accumulator qacc faithfully models the dynamics.

We call a P-argument value *critical* if it has the form $6 \cdot 2^j - 7$ for some $j \geq 1$, i.e., it belongs to $\{5, 17, 41, 89, 185, 377, 761, \ldots\}$. The significance of critical values is that a critical P-argument $6 \cdot 2^j - 7$ is always odd, so rule PQ fires immediately, producing $Q(3 \cdot 2^j - 2, 1)$—a forbidden Q-entry. This section proves the converse: every forbidden Q-entry in the orbit must have been preceded by a critical P-value.

We approach this by backward analysis: which states can produce $Q(1, b)$ as a successor, and how can those predecessors themselves be reached?

Lemma 4.1 (Predecessors of $Q(m, b)$ for $m \geq 2$)**.** *If* $\mathsf{next}(s) = Q(m, b)$ *with* $m \geq 2$*, then exactly one of:*

(a) $s = P(2m - 3)$ *and* $b = 1$ *(via PQ),*

(b) $s = Q(1, 2m - 4)$ *and* $b = 1$ *(via Q1),*

(c) $s = Q(2m + 2, b')$ *for some* b'*, with* $b = b' + 2m + 5$ *(via Q2).*

Proof. Exhaustive case analysis on s. If $s = \mathsf{Halt}$, the successor is Halt, contradiction. If $s = P(a)$, split on the parity of a: even gives $P(\cdots)$, which cannot equal $Q(m, b)$; odd gives $Q(a/2 + 2, 1)$, matching case (a) with $m = a/2 + 2$, i.e., $a = 2m - 3$. If $s = Q(0, b')$, the successor is Halt. If $s = Q(1, b')$, split on parity: even gives $Q(b'/2 + 2, 1)$ matching case (b); odd gives $P(\cdots)$, contradiction. If $s = Q(a' + 2, b')$, split on parity of a': even gives $Q(a'/2, b' + a' + 5)$ matching case (c) with $2m + 2 = a' + 2$; odd gives $P(\cdots)$, contradiction. ☐

Lemma 4.2 (Unique predecessor of $Q(1, b)$)**.** *If* $\mathsf{next}(s) = Q(1, b)$*, then* $s = Q(4, b - 7)$*, i.e., there exists* b' *with* $s = Q(4, b')$ *and* $b = b' + 7$*.*

Proof. By the same case analysis as Lemma 4.1, but specialised to $m = 1$. All cases except $Q(2 \cdot 1 + 2, b') = Q(4, b')$ lead to contradictions: the PQ case would require $P(-1)$, the Q1 case $Q(1, -2)$, and the QP cases produce P-states. ☐

Theorem 4.3 (Forbidden Q-entries require critical P-values)**.** *Let* $k \geq 1$*. If the orbit from* $P(2)$ *reaches a forbidden Q-entry* $Q(3 \cdot 2^k - 2,\, b)$ *for any* b*, then some earlier orbit state was a critical P-value: there exist* $j \geq 1$ *and* n' *such that*

$$\mathsf{next}^{n'}(P(2)) = P(6 \cdot 2^j - 7).$$

Proof. Strong induction on the orbit step n at which $Q(3 \cdot 2^k - 2, b)$ is reached.

For $n = 0$: $\mathsf{next}^0(P(2)) = P(2) \neq Q(\cdots)$, contradiction.

For $n + 1$: by Lemma 1.1, let $s = \mathsf{next}^n(P(2))$ so that $\mathsf{next}(s) = Q(3 \cdot 2^k - 2, b)$. Since $3 \cdot 2^k - 2 \geq 4 \geq 2$ for $k \geq 1$, Lemma 4.1 applies, giving three cases:

(a) **PQ:** $s = P(2(3 \cdot 2^k - 2) - 3) = P(6 \cdot 2^k - 7)$. Take $j = k$.

(b) **Q1:** $s = Q(1,\, 2(3 \cdot 2^k - 2) - 4)$. By Lemma 4.2, the predecessor of this $Q(1, \_)$ at step $n - 1$ is some $Q(4, b'')$. Since $4 = 3 \cdot 2^1 - 2$, the inductive hypothesis applies at step $n - 1$ with $k' = 1$.

(c) **Q2:** $s = Q(2(3 \cdot 2^k - 2) + 2,\, b') = Q(3 \cdot 2^{k+1} - 2,\, b')$. The inductive hypothesis applies at step n with $k' = k + 1$.

In every case we obtain the desired critical P-value. ☐

5 The halving chain (forward direction)

The previous section showed that any forbidden Q-entry in the orbit must have been preceded by a critical P-value. Conversely, we now show that hitting a critical P-value forces the orbit into $Q(1, \_)$: the PQ rule produces a forbidden Q-entry, and the halving chain then inevitably reaches $Q(1, \_)$.

Lemma 5.1 (Halving chain)**.** *For* $k \geq 1$ *and any* b*, there exists* b' *such that*

$$\mathsf{next}^k\big(Q(3 \cdot 2^k - 2,\, b)\big) = Q(1,\, b').$$

Proof. Induction on k.

Base ($k = 1$): $Q(4, b) \xrightarrow{\text{Q2}} Q(1, b + 7)$.

Step ($k + 1$, with $k \geq 1$): We have $3 \cdot 2^{k+1} - 2 = 2(3 \cdot 2^k - 2) + 2$, which is even and ≥ 4, so rule Q2 fires:

$$Q\big(2(3 \cdot 2^k - 2) + 2,\, b\big) \xrightarrow{\text{Q2}} Q\big(3 \cdot 2^k - 2,\, b + 2(3 \cdot 2^k - 2) + 5\big).$$

The inductive hypothesis gives the remaining k steps to $Q(1, \_)$. □

Proposition 5.2 (Critical P-values reach $Q(1, \_)$)**.** *For $k \geq 1$, there exists b such that*

$$\mathsf{next}^{k+1}\big(P(6 \cdot 2^k - 7)\big) = Q(1,\, b).$$

Proof. Since $6 \cdot 2^k - 7$ is odd ($6 \cdot 2^k$ is even), rule PQ fires:

$$P(6 \cdot 2^k - 7) \xrightarrow{\text{PQ}} Q\left(\frac{6 \cdot 2^k - 8}{2} + 2,\ 1\right) = Q(3 \cdot 2^k - 2,\ 1).$$

The halving chain (Lemma 5.1) takes k more steps to $Q(1, b')$. □

6 The main conjecture and its consequences

Theorem 4.3 and Proposition 5.2 together give a complete equivalence:

> *$Q(1, \_)$ is reachable from $P(2)$ if and only if the orbit visits a critical P-value.*

Conjecture 6.1 ($Q(1, \_)$ unreachability)**.** For all $n \in \mathbb{N}$ and $b \in \mathbb{N}$,

$$\mathsf{next}^n(P(2)) \neq Q(1,\, b).$$

Equivalently, rules Q1 and QP1 are never triggered on the orbit from $P(2)$.

Corollary 6.2. *If Conjecture 6.1 holds, then:*

(i) Rule Q1 ($Q(1, 2b) \to Q(b + 2, 1)$) is never reached.

(ii) Rule QP1 ($Q(1, 2b + 1) \to P(3b + 8)$) is never reached.

Proof. Immediate from Conjecture 6.1: if $Q(1, b)$ is never reached, neither rule can fire. □

7 Arithmetic obstructions

Although the conjecture remains open, several arithmetic lemmas constrain how a critical P-value could arise in the orbit. These eliminate all but one mechanism, narrowing the problem to a single Collatz-hard condition.

Lemma 7.1 (PE cannot produce critical P-values)**.** *For all $k, j \in \mathbb{N}$ with $j \geq 1$,*

$$3k + 4 \neq 6 \cdot 2^j - 7.$$

Proof. Modular arithmetic: $3k + 4 \equiv 1 \pmod 3$, while $6 \cdot 2^j - 7 \equiv 0 - 1 \equiv 2 \pmod 3$. □

This means the PE rule $P(2a) \to P(3a + 4)$ can never produce a critical P-value. Since every critical value is odd, the only way one can appear in the orbit is as the output of a Q-phase exit (rule QP1 or QP2).

Lemma 7.2 (Closed-form Q-entries are never forbidden)**.** *For all* $\ell, k, j \in \mathbb{N}$ *with* $k \geq 1$ *and* $j \geq 1$,

$$\frac{(2\ell+1) \cdot 3^k - 5}{2} \neq 3 \cdot 2^j - 2.$$

Proof. Assume equality. Clearing the denominator (both sides are integers since $(2\ell+1) \cdot 3^k$ is odd) gives $(2\ell+1) \cdot 3^k = 6 \cdot 2^j + 1$. But the left side is divisible by 3 (since $k \geq 1$), while $6 \cdot 2^j + 1 \equiv 1 \pmod 3$. □

By Corollary 2.4, we can check case by case whether $p(n)$ can be a forbidden Q-entry $3 \cdot 2^j - 2$. When $v = v_2(n+8) \geq 1$ (i.e., n is even), the Q-entry has the form $((2\ell+1) \cdot 3^v - 5)/2$ with $v \geq 1$, and the lemma above shows it is never forbidden. When $v = 0$ (i.e., n is odd), $p(n) = (n+3)/2$ by Corollary 2.4, which equals $3 \cdot 2^j - 2$ precisely when $n = 6 \cdot 2^j - 7$—a critical P-value. Thus the P-phase iterations never steer the orbit into a forbidden Q-entry; the only way to get one is if the P-argument was already odd and critical upon entry, meaning some prior Q-phase exit produced that critical value.

Lemma 7.3 (QP2 output is critical iff. . .)**.** *The QP2 output* $b+5a+6$ *is a critical P-value* $6 \cdot 2^j - 7$ *(with* $j \geq 1$*) if and only if* $b + 5a + 13 = 6 \cdot 2^j$*.*

Proof. Immediate arithmetic (both sides integral since $6 \cdot 2^j \geq 12$). □

Remark 7.4 (Collatz-hardness)*.* Combining the above:

- PE-outputs $3k + 4$ are never critical (Lemma 7.1).
- Q-entries $p(n)$ with $v_2(n+8) \geq 1$ are never forbidden (Lemma 7.2).
- The only remaining source of critical P-values is the QP2 rule: $Q(2a+3, b) \to P(b + 5a + 6)$.

Thus $Q(1, \_)$ is reachable from $P(2)$ if and only if there exist a reachable state $Q(2a+3, b)$ and $j \geq 1$ with $b + 5a + 13 = 6 \cdot 2^j$, i.e., some QP2 exit produces a critical P-value. No modular invariant (checked computationally up to modulus 96) separates the orbit values from this condition. The problem has the character of a Collatz-type question: the interleaving of multiplicative (times 3) and additive structure in the P-phase, combined with the binary decomposition in the Q-phase, prevents standard density or congruence arguments from closing the proof.

8 Growth analysis of the Q–P–Q cycle

We now analyse the growth rate of Q-entry values across successive Q–P–Q cycles. The central result is that every cycle amplifies the Q-entry, with a minimum growth factor of 5/4.

Q-phase output

When the Q-phase starts at $Q(x, 1)$ with x odd (the most common case), rule QP2 fires immediately:

Lemma 8.1 (Q-phase output for odd entry)**.** *For* $k \geq 0$, $q(2k+1, 1) = 5k + 2$*. Equivalently,* $2\,q(x, 1) = 5x - 1$ *for odd* x*.*

Proof. $q(2k+1, 1) = 1 + \mathrm{qacc}(2k+1) = 1 + 5k + 1 = 5k + 2$. □

When $v_2(x+2) = 1$ (the next most common case), one halving step precedes the QP2 exit:

Lemma 8.2 (Q-phase output for $v_2 = 1$). *For even $k \geq 2$ with $k{+}1$ odd, $2\,q(2k,1) = 9k$.*

Proof. Write $k = 2m$ with $m \geq 1$. Then $\mathrm{qacc}(4m) = 4m{+}3{+}\mathrm{qacc}(2m{-}1) = 4m{+}3{+}(5(m{-}1){+}1) = 9m - 1$, so $q(4m,1) = 1 + 9m - 1 = 9m$ and $2q(4m,1) = 18m = 9 \cdot 2m = 9k$. ☐

More generally, the Q-output ratio $q(x,1)/x$ takes the discrete values $5/2, 9/4, 17/8, 33/16, \ldots$ depending on $v_2(x{+}2)$. From the closed form (Theorem 3.5):

$$q(x,1) = 1 + \mathrm{qacc}(x) = 2x + v_2(x{+}2) + \frac{\max(\mathrm{ord}_2(x{+}2), 3) - 3}{2},$$

so $q(x,1)/x \to 2 + 2^{-(v+1)}$ where $v = v_2(x{+}2)$.

Full-cycle growth factor

The *dominant path* is the case where both the Q-phase and P-phase take the $v_2 = 0$ shortcut.

Proposition 8.3 (Dominant-path cycle). *Let $x = 2k{+}1$ with k odd (i.e., $x \equiv 3 \pmod 4$). Then*

$$p\big(q(x,1)\big) = \frac{5x+5}{4}.$$

The growth ratio $x'/x = (5x{+}5)/(4x) \to 5/4$ as $x \to \infty$.

Proof. By Lemma **??**, $q(x,1) = 5k + 2$. Since k is odd, $5k+2$ is also odd, so $p(5k+2) = (5k+5)/2$ by Lemma 2.2. Then $4 \cdot p(q(x,1)) = 4 \cdot (5k+5)/2 = 2(5k+5) = 5(2k+1) + 5 = 5x + 5$. ☐

Remark 8.4. For the other parity combinations, the growth factor is strictly larger: $v_2 = 1$ gives growth $\approx 15/8 = 1.875$, $v_2 = 2$ gives $\approx 45/16 \approx 2.81$, etc. There is *no contraction* in the Q–P–Q cycle: the P-even rule $P(2a) \to P(3a{+}4)$ amplifies by 3/2 per application, and the Q-phase roughly doubles the input (Lemma **??**).

Hitting condition on the dominant path

For the dominant-path cycle output $y = 5k + 2$ to equal a direct-halting Q-entry $2^i - 2$, we need $5k + 9 = 2^{i+1}$.

Lemma 8.5 (Dominant-path Diophantine condition). *Let k be odd and $i \geq 2$. Then $p(5k+2) = 2^i - 2$ if and only if $5k + 9 = 2^{i+1}$.*

Proof. Since $5k+2$ is odd, apply the $v_2 = 0$ case of Corollary 2.5: $p(n) = 2^i - 2 \iff n + 7 = 2^{i+1}$. Substitute $n = 5k + 2$. ☐

Lemma 8.6 (Mod-5 obstruction). *If $5k{+}9 = 2^{i+1}$, then $i{+}1 \equiv 2 \pmod 4$, i.e., $i \in \{5, 9, 13, 17, \ldots\}$.*

Proof. From $5k + 9 = 2^{i+1}$ we get $2^{i+1} \equiv 9 \equiv 4 \pmod 5$. Since $2^n \bmod 5$ cycles with period 4 as $2, 4, 3, 1, 2, 4, \ldots$ (using $2^4 = 16 \equiv 1 \pmod 5$), the residue 4 occurs precisely when $n \equiv 2 \pmod 4$. ☐

9 Stochastic growth model and hitting density

Since every Q–P–Q cycle grows, the Q-entry values diverge to infinity. We formalise this with a stochastic model that provides quantitative bounds on the probability of hitting specific targets.

The model

We model the 2-adic valuation at each cycle as an independent Bernoulli trial: with probability 1/2 the cycle takes the dominant path ($v_2 = 0$, growth factor 5/4), and with probability 1/2 it takes the $v_2 \geq 1$ path (growth factor $\geq 15/8$).

Formally, let $\Omega = \{0, 1\}^{\mathbb{N}}$ with the product Bernoulli(1/2) measure $\mathbb{P}$. Define the stochastic process

$$
\begin{aligned}
X_0(\omega) &= x_0, \\
X_{n+1}(\omega) &= \begin{cases} (5X_n + 5)/4 & \text{if } \omega_n = 1 \ (v_2 = 0), \\ (15X_n + 25)/8 & \text{if } \omega_n = 0 \ (v_2 \geq 1). \end{cases}
\end{aligned}
$$

Deterministic lower bound

Since both branches dominate the $v_2 = 0$ branch, every path satisfies $X_n(\omega) \geq f^{(n)}(x_0)$ where $f(x) = (5x + 5)/4$.

Lemma 9.1 (Iterate closed form)**.** *The map $f(x) = (5x + 5)/4$ has fixed point $x = -5$ and $f^{(n)}(x_0) = (x_0 + 5)(5/4)^n - 5$.*

Proof. Induction: with $y = x + 5$, $f(x) + 5 = 5(x + 5)/4$, so $y_n = y_0(5/4)^n$. ☐

Proposition 9.2 (Trajectory lower bound)**.** *For $x_0 \geq 0$, every path satisfies*

$$X_n(\omega) \ \geq \ (x_0 + 5) \left(\frac{5}{4}\right)^n - 5.$$

In particular, $X_n \to \infty$ as $n \to \infty$.

Proof. By induction on n. The base case is trivial. For the step, both branches of the stochastic step satisfy $\mathsf{step}(x, b) \geq (5x + 5)/4$ for $x \geq 0$ (since $(15x + 25)/8 \geq (5x + 5)/4 \iff 5x + 5 \geq 0$), and f is monotone increasing. Then apply Lemma **??**. ☐

Expected log-drift

Proposition 9.3 (Positive drift)**.** *The asymptotic per-step expected log-growth is positive:*

$$\frac{1}{2} \ln \frac{5}{4} + \frac{1}{2} \ln \frac{15}{8} = \frac{1}{2} \ln \frac{75}{32} \approx 0.426 > 0.$$

Proof. $\frac{5}{4} \cdot \frac{15}{8} = \frac{75}{32} > 1$, so $\ln(75/32) > 0$ and the half-sum is positive. ☐

This confirms that the system is *purely expansive*—there is no contraction mechanism, unlike classical Collatz-type maps.

First-moment hitting bound

Theorem 9.4 (Reciprocal sum bound)**.** *For $x_0 > 0$,*

$$\sum_{n=0}^{\infty} \frac{1}{x_0 \cdot (5/4)^n} = \frac{5}{x_0}.$$

Proof. Factor out $1/x_0$ and evaluate the geometric series $\sum_{n=0}^{\infty}(4/5)^n = 1/(1-4/5) = 5$. $\square$

Corollary 9.5 (Heuristic hitting density)**.** *Under the heuristic that the orbit values are equidistributed modulo lattice points* $\{(2^m - 9)/5\}$*, the expected number of near-misses from starting value* x_0 *is at most*

$$\sum_{n=0}^{\infty} \frac{1}{X_n \cdot \ln 2} \le \frac{5}{x_0 \ln 2} \approx \frac{7.21}{x_0}.$$

Corollary 9.6 (Parametric bound)**.** *If the machine reaches a Q-entry of order* 10^d *(with* $d \ge 1$*) without halting, then the expected number of future near-misses of any target* $2^i - 2$ *is at most*

$$\frac{5}{10^d \cdot \ln 2} < \frac{8}{10^d}.$$

The last inequality uses $\ln 2 > 5/8$*, which follows from* $e^{5/8} < 2$ *(verified via Taylor remainder: the degree-*4 *Taylor polynomial at* $x = 5/8$ *plus error bound gives* $e^{5/8} < 1.869 < 2$*).*

Remark 9.7 (Concrete numerical values)*.* The BB(6) cryptid simulation reaches Q-entries of order 10^{16} within 200 macro-steps. At that point the parametric bound gives

$$\frac{8}{10^{16}} = 8 \times 10^{-16},$$

an astronomically small expected number of future hits. After 500 steps the Q-entries reach order 10^{40}, giving 8×10^{-40}.

Remark 9.8 (Why an exact bound is Collatz-hard)*.* The first-moment bound above is *heuristic*: it treats the orbit values as pseudo-random with respect to the lattice of powers of 2. Making it rigorous for exact Diophantine hits (proving that the orbit hits at most one power of 2) would require showing that $5f^{(n)}(x_0) + 9 = 2^m$ has at most one solution (n, m). As analysed in Section **??**, the 5-adic/2-adic structure of the iteration causes all local (mod 2^k) constraints to be identically satisfied, and Baker-type bounds on linear forms in logarithms do not yield a contradiction because m grows only logarithmically. This places the exact bound squarely in Collatz-hard territory.

10 On the impossibility of a deterministic hitting bound

We briefly explain why a direct proof that the dominant-path orbit $x_0, f(x_0), f^{(2)}(x_0), \ldots$ (with $f(x) = \lfloor (5x+5)/4 \rfloor$) hits the set $\{(2^m - 9)/5 : m \in \mathbb{N}\}$ at most once appears to be out of reach.

The master equation

Suppose there are two hits at steps $n_1 < n_2$, giving $5f^{(n_i)}(x_0) + 9 = 2^{m_i}$ for $i = 1, 2$. Let $\Delta n = n_2 - n_1$. By unrolling the floor division with truncation errors $e_k \in \{0, 1, 2, 3\}$, one obtains the exact identity

$$4^{\Delta n} X_{n_2} = 5^{\Delta n} X_{n_1} + \sum_{k=0}^{\Delta n - 1} 5^{\Delta n - 1 - k}\, 4^k\, (5 - e_k),$$

and substituting the hit conditions yields

$$2^{m_2 + 2\Delta n} + \text{lower-order} = 5^{\Delta n}(2^{m_1} + 16) + \text{error accumulator}.$$

Failure of 2-adic analysis

The map f has rational fixed point $x = -5$. Setting $Y_n = X_n + 5$, the map without truncation is perfectly multiplicative: $Y_n = Y_0(5/4)^n$. The truncation errors e_k are precisely the carries needed to ensure X_k remains an integer at each step. Because $X_{n_1} = (2^{m_1} - 9)/5$ possesses large 2-adic divisibility, the master equation is automatically balanced modulo every power of 2. No finite 2-adic check can produce a contradiction.

Failure of the Archimedean bound

Dividing the master equation by $5^{\Delta n} 2^{m_1}$ gives

$$\left| 2^{m_2 - m_1} \left(\tfrac{4}{5}\right)^{\Delta n} - 1 \right| = O(2^{-m_1}).$$

Exponentiating: $|(m_2 - m_1)\ln 2 - \Delta n \ln(5/4)| = O(2^{-m_1})$. Since $m_1 \approx \log_2(5x_0) + n_1 \log_2(5/4)$, the right side is $O(1/x_0)$ when $n_1 = 0$. By Baker's theorem, the left side is at least $C\,\Delta n^{-\kappa}$ for effective constants C, κ. But for $\Delta n \sim x_0$, this lower bound is also $O(1/x_0)$, so the two sides can match and no contradiction arises.

Conclusion

Both the local (2-adic) and global (Archimedean) approaches fail. The deterministic hitting bound is therefore Collatz-hard, and we rely on the heuristic first-moment bound of the previous section.