1. Matrix formulation of BMO1

BMO1 can be stated as the following question:

Start with $(a_0, b_0) = (1, 2)$, and iterate it under the following map:

$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n - b_n, 4b_n + 2) \text{ if } a_n > b_n \\ (2a_n + 1, b_n - a_n) \text{ if } a_n < b_n \end{cases}$$

Does $a_n = b_n$ for some n?

We can rewrite the iteration in a matrix form:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \vec{a}_n = \begin{pmatrix} a_n \\ b_n \\ 1 \end{pmatrix}$$

$$\vec{a}_{n+1} = \begin{cases} A\vec{a}_n \text{ if } a_n > b_n \\ B\vec{a}_n \text{ if } a_n < b_n \end{cases}$$

I claim BMO1 is equivalent to the following question:

Q1. Does there exist a finite product P of A and B such that

$$\begin{pmatrix} -1 & 1 & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0?$$

Proof: If BMO1 halts, then P representing the actual evolution of the machine satisfies the above equation. If some P satisfies the above equation, either it represents the true evolution of the machine, or it takes a "wrong" step somewhere. It can be checked that this will cause a_n and b_n to have opposite signs, after which no further multiplications by A or B can change their signs again. Therefore, a_n and b_n can never be equal if P takes a wrong step. □

2. Zero-in-the-corner form

Define the following:

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, A' = X^{-1}AX, B' = X^{-1}BX$$

I claim the following question is equivalent to Q1:

Q2: Does there exist a finite product P' of A' and B' such that

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix} P' \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} = 0?$$

Proof: Notice that $\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} = X^{-1} \begin{pmatrix}1 \\ 2 \\ 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0 & 0\end{pmatrix} = \begin{pmatrix}-1 & 1 & 0\end{pmatrix} X$. The natural mapping between P and P' shows the equivalence. □

Q2 is also equivalent to asking: can P' have a 0 in its upper-left corner? This is known as a "zero-in-the-corner" problem. Such problems are known to be undecidable in the general case, but it is unknown if it is decidable with 2 3x3 matrices.[1]

For concreteness, here are A' and B' given explicitly:

$$A' = \begin{pmatrix}11 & 4 & 2 \\ -12 & -4 & -2 \\ -10 & -4 & -1\end{pmatrix}, B' = \begin{pmatrix}-2 & -2 & -1 \\ 5 & 4 & 2 \\ 3 & 2 & 2\end{pmatrix}$$

[1] https://arxiv.org/pdf/1404.0644