BB 2x5 Dekaheptoid Non-Halt Proof

dyuan01

July 2024

1 Introduction

We will be looking at the TM 1RB3RB1LB—2RB_2LA1RA4LB2LA2RA, a 2-state 5-symbol TM whose behavior is similar to that of Skelet 17. Another 2-state 5-symbol TM, 1RB3RA2LB1LB1RB_2LA2RA4LA1LA—, also has similar behavior. We will not be going over that TM, but it is claimed that there is an isomorphism between that TM and the TM we will discuss.

2 Figuring out the rules

This is currently a placeholder. It is assumed that the rules mentioned in the next section are accurate.

Define such a list of numbers as S. Let S_i be the i-th index of S starting with index 0.

3 Proof that 1RB3RB1LB—2RB_2LA1RA4LB2LA2RA doesn't halt

From the previous section (currently assumed), we have the following sequence of list of numbers:

1. Start with $[1, 1]$
2. Given a list of numbers S, define the next term of the sequence $P(S)$ as follows (assume S has at least two elements)
   - Halve: If the first term is 0, delete it and increment the new first term by 3 (Example: [0, 18, 9, 4, 1] becomes [21, 9, 4, 1])
   - Even Increment: If the first term is a positive even number, decrement the first term by 1 and increment the second term by 1 (Example, [40, 20, 11, 5, 1] becomes [39, 21, 11, 5, 1])
   - Halt: If the first term is an odd number, all other terms are even numbers, and the last term is 0, then Halt.

- Empty: If the first term is an odd number, all other terms are even numbers, and the last term is non-zero, then decrement the first number and append a 0 to the end of the list (Example: [9, 6, 4, 2] becomes [8, 6, 4, 2, 0])
- Overflow: If the first and last term are odd numbers, and all other terms are even numbers, then decrement the first number and append a 1 to the end of the list (Example: [1, 10, 4, 1] becomes [0, 10, 4, 1, 1])
- Odd Increment: Otherwise (the first term is odd and there is another term before the last term that is also odd), consider the leftmost odd number that is not the first term. Increment the number immediately to its right by 1, and decrement the first term by 1. (Examples: [5, 8, 3, 2, 0] becomes [4, 8, 3, 3, 0]; [7, 9, 4, 2, 0] becomes [6, 9, 5, 2, 0]; [3, 14, 6, 3, 0] becomes [2, 14, 6, 3, 1])

Definitions:

Consider two lists of numbers S and T. We say T is a *normal successor* of S if you can get T by incrementing one of the elements in S by 2, and we say T is an *overflow successor* of S if you can get T by appending 0 to the ends of S then incrementing one of the elements in S by 2.

We define a list of numbers S to be *secure* if one of two things is true:

1. Type 1: There exists a normal successor of S that can be reached by repeatedly applying $P(S)$ to it. And there's exactly 1 Halve rule, 1 Empty rule, and 0 Overflow/Halt rules applied in the process.

2. Type 2: There exists an overflow successor of S that can be reached by repeatedly applying $P(S)$ to it, the successor's last element is non-zero, and while applying $P(S)$, you encounter exactly 1 Empty rule, 1 Overflow rule, and 1 Halve rule in that order, with no other instances of these rules or halting rules.

3. Type 3: There exists an overflow successor of S that can be reached by repeatedly applying $P(S)$ to it, and while applying $P(S)$, you encounter exactly 1 Overflow rule, 1 Halve rule, and one Empty rule in that order, with no other instances of these rules or halting rules.

Theorem 1: If S is Type 1 secure, then $P(S)$ is secure (any type).

Proof: Let T be the normal successor of S from the definition of Type 1 secure.

We have different cases on S

- If the transition from S to $P(S)$ is an Even Increment rule, that would make T to $P(T)$ also an Even Increment rule. It follows that $P(T)$ is a normal successor of $P(S)$, additionally, the number of Empty, Zero, Overflow, and Halt rules are the same, so $P(S)$ is Type 1 secure.

- If the transition from S to $P(S)$ is an Odd Increment rule, that would also make T to $P(T)$ an odd increment rule. Since the index of the leftmost odd term is the same between S and T, it follows that $P(T)$ is a normal successor of $P(S)$. Similar to before, the number of each rule type doesn't change, so $P(S)$ is Type 1 secure.

- If the transition from S to $P(S)$ is an Empty rule, then so is the transition from T to $P(T)$. Since S and T are the same length, appending a 0 to the end is the same for both sides. It follows that $P(T)$ is also a normal successor of $P(S)$. The empty rule is removed from S to $P(S)$, but added back from T to $P(T)$ so $P(S)$ is Type 1 secure.

- If the transition from S to $P(S)$ is a Halve rule, then from the lemma, we know that T is a normal successor of S. There are three cases:

  1. If T to $P(T)$ is also a halve rule, then $P(T)$ is a normal successor of $P(S)$. This means $P(S)$ is Type 1 secure.
  2. Otherwise, the first element of T is 2. (I got lazy, but basically, T starts with 2, so if T does an even then odd increment, halve, then do another increment, it will end up being a normal successor of $P(S)$, making $P(S)$ Type 1 secure)
  3. (Also got lazy here, but if T does an even increment, overflow, halve, then another increment, you get an overflow successor, making $P(S)$ type 2 secure)

Since in every case, we can find another successor for $P(S)$, we have completed the proof.

Theorem 2: If S is Type 2 secure, then $P(S)$ is secure (any type).

Proof: Let T be the overflow successor of S from the definition of Type 2 secure.

We have different cases on S

- If the transition from S to $P(S)$ is an Even Increment rule, that would make T to $P(T)$ also an Even Increment rule. It follows that $P(T)$ is an overflow successor of $P(S)$, so $P(S)$ is Type 2 secure.

- If the transition from S to $P(S)$ is an Odd Increment rule, that would also make T to $P(T)$ an Odd Increment rule. Since the index of the leftmost odd term is the same between S and T, it follows that $P(T)$ is an overflow successor of $P(S)$. So $P(S)$ is Type 2 secure.

- If the transition from S to $P(S)$ is an Empty rule, then so is the transition from T to $P(T)$. (It's important to note that the last element of T is non-zero). It follows that $P(T)$ is also an overflow successor of $P(S)$. However, since the Empty rule is now after the Overflow and Halve rule, $P(S)$ becomes Type 3 secure.

Since in every case, we can find another successor for $P(S)$, we have completed the proof.

Theorem 3: If S is Type 3 secure, then $P(S)$ is secure (any type).

Proof: Let T be the overflow successor of S from the definition of Type 3 secure.

We have different cases on S

- If the transition from S to $P(S)$ is an Even Increment rule, that would make T to $P(T)$ also an Even Increment rule. It follows that $P(T)$ is an overflow successor of $P(S)$, so $P(S)$ is Type 3 secure.

- If the transition from S to $P(S)$ is an Odd Increment rule, that would also make T to $P(T)$ an Odd Increment rule. Since the index of the leftmost odd term is the same between S and T, it follows that $P(T)$ is an overflow successor of $P(S)$. So $P(S)$ is Type 3 secure.

- If the transition from S to $P(S)$ is an overflow rule, then we know that T is an overflow successor of S. It follows that T to $P(T)$ is an Odd Increment rule. It follows that $P(T)$ is a successor of $P(S)$. And that $P(S)$ is Type 1 secure.

Since in every case, we can find another successor for $P(S)$, we have completed the proof.

This means if S is secure, then $P(S)$ is secure. Since $[1, 1]$ is secure, every term after that is secure, and also can't halt, so this sequence never halts.