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Theorem 1. *Let i, k be positive integers such that $i \geq 2^{2100}$ and*

$$\frac{3^i}{2i} \leq 2^k.$$

Then,

$$2 \cdot 3^i + i + 5 \neq 0 \mod 2^k.$$

To prove Theorem 1, we make use of a theorem from Ellison [1970-1971], which we will introduce below.

Theorem 2 (Ellison [1970-1971], Corollary 1)**.** *Let a, b, m, and n be positive integers. If $m \leq n$ and δ is a given positive real number, then for all positive integers $x \geq x_0$ and y we have*

$$|am^x - bn^y| \geq m^{(1-\delta)x} \quad or \quad |am^x - bn^y| = 0,$$

where

$$x_0 = \left(\frac{2^{31}\log A}{\Delta}\right)^{49}, \quad A = \max(4, a, b, m, n), \quad \Delta = \min(2, \delta \log m, \delta \log n).$$

We can now proceed with the proof of Theorem 1.

Proof. Let c be a positive integer. Then, since

$$\forall c \left|2 \cdot 3^i - c \cdot 2^k\right| > i + 5 \implies 2 \cdot 3^i + i + 5 \neq 0 \mod 2^k,$$

it suffices to prove that $2 \cdot 3^i$ cannot be too close to a multiple of 2^k. Since

$$\frac{3^i}{2i} \leq 2^k \implies 2 \cdot 3^i \leq 4i \cdot 2^k,$$

it suffices to consider $c \leq 4i$. Otherwise, all $c > 4i$ are guaranteed to work because

$$\left|2 \cdot 3^i - c \cdot 2^k\right| \geq 2^k \geq \frac{3^i}{2i} > i + 5$$

for large enough i. Putting our expression in the form used by Theorem 2, we have

$$\left|2 \cdot 3^i - c \cdot 2^k\right| = \left|c \cdot 2^k - 2 \cdot 3^i\right| = |am^x + bn^y| \implies (a, b, m, n) = (c, 2, 2, 3).$$

It is easy to see that $\left|2 \cdot 3^i - c \cdot 2^k\right|$ cannot ever be equal to zero for large i because $\nu_2(2 \cdot 3^i) = 1 < k \leq \nu_2(c \cdot 2^k)$ for large enough i. Then, if we arbitrarily select $\delta = 0.9$, Theorem 2 forces

$$\left|c \cdot 2^k - 2 \cdot 3^i\right| \geq 2^{0.1k}$$

for $k \geq x_0$. Since $c \leq 4i$, we must have

$$A = \max(4, a, b, m, n) = \max(4, c, 2, 2, 3) \leq 4i,$$

and

$$\Delta = \min(2, \delta \log m, \delta \log n) = \min(2, 0.9 \log 2, 0.9 \log 3) = 0.9 \log 2 > 0.6.$$

Then,

$$x_0 = \left(\frac{2^{31}\log A}{\Delta}\right)^{49} \le \left(\frac{2^{31}\log(4i)}{0.6}\right)^{49} < 2^{1556}\log(4i)^{49}.$$

If we set $i = 2^{2100}$, then

$$k \ge \log_2\left(\frac{3^i}{2i}\right) = i\log_2(3) - \log_2(2i) > 2^{2100} > 2^{1556}\log_2(4i)^{49} > \frac{2^{1556}\log_2(4i)^{49}}{\log_2(e)^{49}} = 2^{1556}\log(4i)^{49},$$

so $k \ge x_0$ if $i = 2^{2100}$. We will then show that $k \ge x_0$ for all $i \ge 2^{2100}$ by comparing derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}i} 2^{1556}\log(4i)^{49} = \frac{49 \cdot 2^{1556}\log(4i)^{48}}{i} < \frac{2^{1562}\log(4i)^{48}}{i}.$$

The RHS is decreasing for $i > 1.8 \cdot 10^{20}$, and is slightly less than 0.0035 when $i = 2^{2100}$. Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}i} 2^{1556}\log(4i)^{49}\right|_{i \ge 2^{2100}} < \left.\frac{2^{1562}\log(4i)^{48}}{i}\right|_{i \ge 2^{2100}} < 0.0035,$$

and

$$0.0035 < \log_2(3) - 2^{-2100} < \log_2(3) - \left.\frac{1}{i\log 2}\right|_{i \ge 2^{2100}} = \left.\frac{\mathrm{d}}{\mathrm{d}i}\log_2\left(\frac{3^i}{2i}\right)\right|_{i \ge 2^{2100}}.$$

So, if $i \ge 2^{2100}$, then $k \ge x_0$ and

$$\left|c \cdot 2^k - 2 \cdot 3^i\right| \ge 2^{0.1k}$$

holds. It then remains to show that the RHS of $2^{0.1k}$ is greater than $i + 5$. Thankfully, this is easy:

$$2^{0.1k} \ge 2^{0.1 \cdot \log_2\left(3^i/(2i)\right)} = \left(\frac{3^i}{2i}\right)^{0.1} > \frac{1.1^i}{2i},$$

which is greater than $i + 5$ as long as $i \ge 106$. □

Finally, we present a corollary of Theorem 1 that proves the desired result.

Corollary 3. *Let i be a positive integer such that $i \ge 2^{2100}$. Then,*

$$\frac{2 \cdot 3^i + i + 5}{2^{\nu_2(2 \cdot 3^i + i + 5)}} \ge 2i + 14.$$

Proof. By Theorem 1, 2^k cannot divide $2 \cdot 3^i + i + 5$ where $2^k \le 3^i/(2i)$, so

$$\begin{aligned}
\nu_2(2 \cdot 3^i + i + 5) &< k \\
2^{\nu_2(2 \cdot 3^i + i + 5)} &< 2^k \le \frac{3^i}{2i} \\
2^{\nu_2(2 \cdot 3^i + i + 5)} &< \frac{2 \cdot 3^i}{4i} \\
2^{\nu_2(2 \cdot 3^i + i + 5)} &< \frac{2 \cdot 3^i + i + 5}{4i} \\
2^{\nu_2(2 \cdot 3^i + i + 5)} &< \frac{2 \cdot 3^i + i + 5}{2i + 14} \\
2i + 14 &< \frac{2 \cdot 3^i + i + 5}{2^{\nu_2(2 \cdot 3^i + i + 5)}}.
\end{aligned}$$

□

References

W. J. Ellison. On a theorem of s. sivasankaranarayana pillai. *Séminaire de théorie des nombres de Bordeaux*, pages 1–10, 1970-1971. URL `http://eudml.org/doc/275222`.